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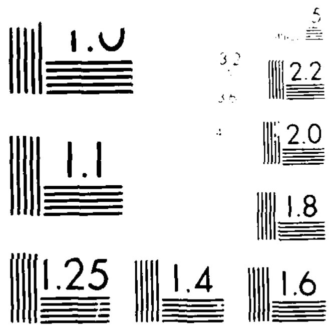
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# A CLASS OF STABLE TRANSMISSION ALGORITHMS FOR VARYING USER MODELS

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**Department of**

**Electrical Engineering and Computer Science**

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A CLASS OF STABLE TRANSMISSION ALGORITHMS FOR VARYING USER MODELS

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ABSTRACT

We propose and analyze a class of stable transmission algorithms whose operation is independent of the number of users in the system and the arrival process per user, as long as the latter is i.i.d. The algorithms in the class are a combination of a random access and a reservation techniques, they are synchronous, and they are studied in the full sensing broadcast environment. For any finite number of independent users in the system, and any i.i.d. arrival process per user, their throughput is one. Given some i.i.d. arrival process per user, given some algorithm in the class, its limit throughput, when the user population tends to infinity, is lower bounded by its throughput in the presence of the limit Poisson user model. The latter throughput is also attainable; it coincides with the limit throughput, when the users are Poisson. Due to the above, it is concluded that the limit Poisson user model is an indispensable vehicle in the study of the algorithms in the class.

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## 1. Introduction

We consider a broadcast packet radio network with independent and identical users. We require that the time of the transmission channel be slotted, and that transmissions be then synchronous (each packet transmission may only start at the beginning of some slot). We assume ternary feedback per slot (empty versus success versus collision), full feedback sensing by each user, and no propagation delays. We also assume that a collision results in full destruction of all the involved packets; thus, retransmission is then necessary. Given the above general description of the overall system, we wish to provide multiplexing techniques, whose operation is independent of the number uf users in the network and of the arrival process per user (assuming that the latter is i.i.d.), whose stability is guaranteed even when the number of users tends to infinity, and whose delay characteristics are uniformly good within their stability region. In this paper, we propose and analyze a general class of such multiplexing algorithms, whose members attain throughput one for any finite number of users, and are stable when the number of users tends to infinity. An important part of our work corresponds to a concrete justification of the limit Poisson user model, as an indispensable vehicle in the study of random access algorithms.

We assume that each one of the independent and identical users in the system has the following characteristics; (i) His packet generating process is arbitrary i.i.d., with mean $\lambda$ packets/slot, and finite second moment. (ii) The user possesses a buffer, where he stores his nontransmitted packets on the first come-first serve basis. The earliest stored arrival lies on the head of the buffer queue, and is called the head packet in the queue.

Time will be measured in slot units. The integers, T and t, will denote slot indices, where slot T occupies the transmission interval (T,T+1]. The ternary feedback corresponding to slot T, will be denoted $x_T$, where $x_T=0$, $x_T=1$, and $x_T=c$, represent respectively empty, versus success (busy with a single packet), versus collision slot T.

We will assume that the system starts operating at time zero; that is, at time zero the queue of each user is empty. A user is called active at time T, if his buffer queue is nonempty at the beginning of slot T; that is, some arrivals in (0,T] are then stored in the queue. It will be also assumed that at each time T, each user knows the overall feedback history, $x_t$; $0 \leq t \leq T-1$.

## 2. The Class of Algorithms

Let the queues of all users be empty at the beginning of slot T-1. Then, at time T each active user transmits his head packet, and a collision resolution process begins. This process involves only the head packets of the active at T users, and the initial contention among them is resolved via some random access algorithm (RAA), whose general characteristics will be described below. At the time T above, a collision resolution interval (CRI) starts. During its progress the users who were inactive at T withhold transmissions, and only the packets that were queued at T are transmitted. The CRI ends at the time $T'$, when it is known by all users that all the latter packets are successfully transmitted. Then, at time $T'+1$ the next CRI begins, with the transmission of the head packets of all the active at $T'+1$ users.

Given some RAA, let T correspond to the starting point of some CRI. Then, the algorithm operates as follows. Let t be some slot within the CRI, such that $x_t=1$ and $x_{t-1}\neq 1$. Then, the successful at t user reserves the channel, to sequentially transmit all the packets which were stored in his queue at the beginning of slot T, while the remaining users withhold transmissions. After all the above packets are successfully transmitted, an empty slot, say $t'$, is allowed. Upon observing $x_{t'}=0$, all users in the system know that the transmissions by the successful at t user have ended. Then, at time $t'+1$, the collision resolution process among the head packets is picked up again at the point where it was left (by resetting $t'=t$ and $x_{t'}=x_t=1$), and continues as dictated by the RAA. As claimed before, the CRI clearly ends with the successful transmission of all the packets that were queued at time T. It is also clear that if the CRI ends at $T'$, then $x_{T'}=0$ and $x_{T'-1}=1$. Regarding the RAA used by the algorithm,

we require that it be stable, and it possess the following general characteristics:

Let us assume that at some time instant t, the arrival interval $(0,t']$ ; $t'<t$ has just been resolved by the pure RAA. Then, the RAA examines next some new arrival interval $(t',t'']$, by first allowing all arrivals in it to transmit in slot t+1, and resolves it completely before examining a new interval. Let $\ell_{k,d}$ denote the number of slots needed by the RAA to resolve the latter interval, given that its length is d and that the number of arrivals in it is k. Let $\ell_k$ denote the number of slots needed to resolve a collision of multiplicity k. Then, for $\underset{st}{\leq}$ denoting stochastic dominance, the RAA is such that,

$$E\{\ell_{k,d}\} = E\{\ell_k\} \triangleq L_k \quad , \quad E\{\ell^2_{k,d}\} = E\{\ell^2_k\} \triangleq L^{(2)}_k$$

$$\overline{\lim_{k\to\infty}}\, k^{-2} L^{(2)}_k \leq [\overline{\lim_{k\to\infty}}\, k^{-1} L_k]^2 + 1 < \infty \tag{1.a}$$

$$\ell_k \underset{st}{\leq} \ell_{k+1} \;;\; \forall\, k \tag{1.b}$$

We point out that the above conditions are satisfied by most stable RAAs. A simple such RAA is Capetanakis' algorithm [1], both in its nondynamic and dynamic forms. We note that in the presence of some RAA in the class considered here, the earlier description of a CRI is consistent.

## 3. Analysis-The Finite User Population

We now proceed with the analysis of our class of algorithms, for the finite user population, and the user model described in the introduction. The limit analysis, when the user population tends to infinity, will be presented in section 4.

Consider one of the algorithms in the class (given some RAA as in section 2). Let the system start operating at time zero, and let us consider the sequence (in time) of the CRIs that are generally induced by the algorithm. Let $C_i$ denote the length of the i-th CRI, where $i\geq 1$. Then, the first CRI corresponds to the empty slot zero; thus, $C_1=1$. In addition, the sequence $\{C_i \;;\; 1\leq i<\infty\}$ is clearly a Markov

chain. Let $L_k$ be as in (1), and let us also define the following quantities.

$$L_{i,d} \triangleq E\{C_{i+1} \mid C_i=d\} \qquad (2)$$

$p_i(k|d)$ : The probability that the number of active users at the beginning of the (i+1)-th CRI is k, given that $C_i=d$.

Let us assume that the number of users in the system is $M<\infty$, and let the packet generating process per user be i.i.d., with mean $\lambda$ packets/slot and finite second moment. Let $p_0$ denote the probability of zero arrivals within a single slot time interval, as induced by the above process. Then, we easily conclude,

$$L_d \triangleq L_{i,d} = \sum_{k=0}^{M} L_k\, p_i(k|d) + \lambda d\, [1-p_0^d]^{-1} \sum_{k=0}^{M} k\, p_i(k|d) =$$

$$= \sum_{k=0}^{M} L_k\, p(k|d) + M\lambda d \qquad (3)$$

; where,

$$p(k|d) = p_i(k|d) = \binom{M}{k} p_0^{d(M-k)}\, [1-p_0^d]^k \qquad (4)$$

We now express a theorem, whose proof is in the appendix.

Theorem 1

Given $M<\infty$, the sequence $\{C_i\}_{i\geq 1}$ of CRI lengths induced by the algorithm is clearly a Markov chain. We make it irreducible and aperiodic, by properly choosing its state space. If $M\lambda<1$, then the chain is also ergodic.

---

Let us now consider the packet arrivals in the system, as they evolve in time. Let $\mathcal{D}_n$ denote the delay experienced by the n-th packet arrival, as induced by the algorithm; that is, the time between the arrival of the packet and its successful transmission. Let the sequence $\{T_i\}_{i\geq 1}$ be defined as follows: Each $T_i$ corresponds to the beginning of some slot, and $T_1=1$. In addition, each $T_i$ corresponds to the ending point of a length-one CRI. $T_{i+1}$ is then the ending point of the first after

$T_i$ unity length CRI.

Let $R_i$ ; $i \geq 0$ denote the number of successfully transmitted packets in the time interval $(0, T_{i+1}]$. Then, $A_i \triangleq R_i - R_{i-1}$ ; $i \geq 1$ denotes the number of successfully transmitted packets in the time interval $(T_i, T_{i+1}]$, where $R_0 = 0$. The sequence $\{A_i\}_{i \geq 1}$ is clearly a sequence of i.i.d. random variables; thus, $\{R_i\}_{i \geq 0}$ is a renewal process. In addition, the delay process $\{\mathcal{D}_n\}_{n \geq 1}$ induced by the algorithm is regenerative with respect to the process $\{R_i\}_{i \geq 0}$, and the distribution of $A_i$ is nonperiodic, for each i. The regenerative theorem in [2] then applies, in the following form:

If,

$$A \triangleq E\{A_1\} < \infty, \text{ and } W \triangleq E\{\sum_{i=1}^{A_1} \mathcal{D}_i\} < \infty \tag{5}$$

then, $\mathcal{D}_i$ converges in distribution to a random variable $\mathcal{D}_\infty$, and,

$$D \triangleq E\{\mathcal{D}_\infty\} = WA^{-1} \tag{6}$$

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \mathcal{D}_i = \lim_{n\to\infty} n^{-1} E\left\{\sum_{i=1}^{n} \mathcal{D}_i\right\} = WA^{-1} \;;\; \text{w.p. } 1 \tag{7}$$

We note that (7) holds even if the second moment of the packet generating process per user is not finite. The latter property is necessary, however, for the satisfaction of expression (6).

Due to the above, if we compute the expected values A and W in (5), and show that they are both finite, then, we can guarantee convergence (in distribution) of the delay $\mathcal{D}_i$ ; that is, existence of steady state. In addition, we can then compute the expected per packet delay, as $D = W A^{-1}$. Note that, given M and $\lambda$, we have,

$$A = M\lambda H \quad , \text{ where } H \triangleq E\{T_{i+1} - T_i\} \;;\; i \geq 1 \tag{8}$$

We now express a theorem whose proof is in the appendix.

Theorem 2

Given $M<\infty$, let the packet generating process per user be i.i.d., with finite second moment, and mean $\lambda$. Let also $M\lambda<1$. Then, the expected values A and W are both bounded, and so is then the expected per packet delay D.

We note that given some specific RAA as in section 2, upper and lower bounds on the expected per packet delay D can be computed, via methods as in [2].

Remark

Given M and $\lambda$, the quantity $\lambda_M \triangleq M\lambda$ is called the <u>input rate</u> of the system. Given $M,\lambda$, and some multiplexing algorithm, let us define, $\rho_M \triangleq n^{-1} \sum_{i=1}^{n} I_n$, where $I_n$ equals 1 if the n-th slot is a success slot and it equals zero otherwise. Then, $\rho_M$ is called the <u>output rate</u> of the system. Let $\mathcal{D}_i$ be the delay of the i-th successfully transmitted packet, as induced by the adopted multiplexing technique, and let $\mathcal{D}_{s,M}$ be the steady state delay. Given $M,\lambda$, and some multiplexing technique, let us define:

$$\lambda_M^{\star}(1) \triangleq \sup(\lambda_M : \lambda_M = \rho_M)$$

$$\lambda_M^{\star}(2) \triangleq \sup(\lambda_M : \mathcal{D}_{s,M} < \infty \text{ , w.p.1}) \tag{9}$$

$$\lambda_M^{\star}(3) \triangleq \sup(\lambda_M : E\{\mathcal{D}_{s,M}\} < \infty)$$

To this point,we have shown that if the multiplexing technique used is one of our algorithms, and if the packet generating process per user is arbitrary i.i.d., then given any M we have,

$$\lambda_M^{\star}(1) = \lambda_M^{\star}(2) = 1 \tag{10}$$

If in addition the second moment of the packet generating process per user is finite, then,

$$\lambda_M^{\star}(1) = \lambda_M^{\star}(2) = \lambda_M^{\star}(3) = 1 \tag{11}$$

Thus, for either one of the quantities in (9) being used as measure of throughput, it is concluded that, for any number M of users, and any i.i.d. packet generating process per user that possesses finite second moment, the throughput of every algorithm in our class is one. We can also then conclude,

$$\overline{\lim_{M\to\infty}}\ \lambda_M^*(1) = \overline{\lim_{M\to\infty}}\ \lambda_M^*(2) = \overline{\lim_{M\to\infty}}\ \lambda_M^*(3) = 1 \tag{12}$$

We note that the quantity $\lambda_M^*(3)$ in (9) represents the strongest definition of throughput, and it is perhaps the most meaningful, since it implies finite expected per packet delays. It is very important to point out that,

$$\overline{\lim_{M\to\infty}}\ \lambda_M^*(3) = \overline{\lim_{M\to\infty}}\ \sup(\lambda_M : E\{\mathcal{D}_{s,M}\} < \infty) > \lambda_c^* \tag{13}$$

; where, for $\lambda_c$ denoting system input rate, we define,

$$\lambda_c^* \triangleq \sup(\lambda_c : \overline{\lim_{M\to\infty}}\ E\{\mathcal{D}_{s,M}\} < \infty) \tag{14}$$

Therefore, $\overline{\lim_{M\to\infty}}\ \lambda_M^*(3) = 1$, does not imply $\lambda_c^* = 1$. Yet, $\lambda_c^*$ represents the meaningful definition of throughput in the limit where the user population tends to infinity, since it is then implied that the limit expected per packet delay is finite. We will name $\lambda_c^*$, the <u>limit throughput</u> of the algorithm, at the specific arrival process per user.

## 4. The Limit User Population

In this section, we analyze our algorithms in the limit, where the user population tends to infinity. We maintain the user model presented in the introduction, and we denote by $\lambda$ and $\sigma^2$ the mean and the second moment of the per user i.i.d. packet generating process. As in section 3, we denote by $p_0$ the probability of zero arrivals per slot, as induced by the latter process. Given $M$ users in the system, let us denote by $p_M(k)$ the probability that k previously inactive users become active within a single slot time interval. Then,

$$p_M(k) = \binom{M}{k} p_0^{M-k}(1-p_0)^k \tag{15}$$

Let us now allow $M$ increase to infinity, and $\lambda$ simultaneously decrease, in the following way, for some given $\delta$ in (0,1):

$$\lim_{M\to\infty} M\lambda = \delta \tag{16}$$

The following theorem is basically the Poisson theorem [4], where $p_0$ increases with increasing $M$.

Theorem 3

(i) If $\lim_{M\to\infty} M(1-p_0) = c < \infty$, then $\lim_{M\to\infty} p_M(k) = e^{-c}\,\frac{c^k}{k!}$

If the limit $\lim_{M\to\infty} M(1-p_0)$ does not exist, neither does the limit $\lim_{M\to\infty} p_M(k)$.

(ii) If in addition to (16), it is also required that the mean of the limit distribution be equal to the limit of the mean, then $\delta = c$.

---

Theorem 3 states that independently of the arrival process per user (as long as it is i.i.d.),if it exists, the limit of the cumulative arrival process is Poisson. If for some given $\delta$, the conditions in part (ii) of the rheorem are imposed, then the intensity of this Poisson process is $\delta$.

Let now CRAR be the name of some algorithm within the class described in section 2. Let this algorithm attain throughput $\lambda_\infty^\star$ (by every definition in (9)), in the presence of the limit Poisson user model (infinitely many independent and identical Bernoulli users). For $L_k$ as in (1), let us define:

$$c \triangleq \overline{\lim_{k\to\infty}}\, k^{-1} L_k \;, \text{ and } \eta \triangleq \underline{\lim_{k\to\infty}}\, k^{-1} L_k$$
$$\lambda_1^\star \triangleq (1+c)^{-1} \;, \text{ and } \lambda_2^\star \triangleq (1+\eta)^{-1} \tag{17}$$

; where, as well known,

$$\lambda_1^\star < \lambda_\infty^\star < \lambda_2^\star$$

The main result of this section is then stated in the following theorem, whose proof is presented in subsection 4.1 below.

Theorem 4

Consider the user model in section 1 and in the beginning of this section, where [illegible] c. Let the CRAR be used, and let $D_{s,M}$ be then defined as in section 3. Then,

(i) If, as the number $M$ of users increases, the quantities $\lambda$ and $\sigma^2$ decrease so that,

$$\overline{\lim_{M\to\infty}} M\lambda = \delta < \lambda_1^* \quad , \quad \overline{\lim_{M\to\infty}} M\sigma^2 < \infty$$

then,

$$\overline{\lim_{M\to\infty}} E\{D_{s,M}\} < \infty$$

(ii) Given the CRAR, and given Poisson users, let $M\lambda = \delta$ ; $\forall M$. Let also $\delta > \lambda_2^*$. Then,

$$\overline{\lim_{M\to\infty}} E\{D_{s,M}\} = \infty$$

---

We note that the condition $\ell_k \underset{st}{\leq} \ell_{k+1}$ in (1) is only used in part (ii) of the theorem.

Remark

Due to theorems 3 and 4, the following important conclusions are drawn:

1. The limit Poisson user model is valid. For any i.i.d. arrival process per user, it represents the only possible limit of the cumulative arrival process.
2. Given some algorithm within the class in section 2, let $\lambda_\infty^*$ be its throughput in the presence of the limit Poisson user model. Let $\lambda_\infty^*$ be known with uncertainty; that is, $\lambda_\infty^* \varepsilon (\varepsilon_1, \varepsilon_2)$, for some constants $\varepsilon_1 < \varepsilon_2$. Then, given any i.i.d. arrival process per user, given that the second moment of the cumulative input process is finite, the limit throughput $\lambda_c^*$ of the algorithm is lower bounded by $\varepsilon_1$. Furthermore, if the users are Poisson, then, $\varepsilon_1 < \lambda_c^* < \varepsilon_2$. For many of the known algorithms, $\varepsilon_2 - \varepsilon_1$ is very close to zero.

The above conclusions provide a concrete justification of the limit Poisson user model, as an important vehicle in the study of our algorithms. If in the presence of the above user model, the throughput of some algorithm in our class is $\lambda^*$, then, for any cumulative input rate in $(0, \lambda^*)$, stability and good delay characteristics are guaranteed, independently of the number of users in the system, and independently of the specific i.i.d. arrival process per user.

## 4.1 Proof of Theorem 4

Consider the user model in the introduction, and some algorithm in our class, whose throughput in the presence of the limit Poisson user model is $\lambda^*$. Given $M$

users in the system, let $\pi_{d,M}$ be the steady state probability that the Markov chain $\{C_i\}_{i\geq 1}$ of CRI lengths is in state d, and let $\ell_{s,M}$ be then the steady state CRI length. Let $\sigma^2$ be as in part (i) of theorem 4. Given M, and given that the algorithm starts at the end of a length-d CRI, let $H_{d,M}$ denote the expected number of slots needed to the end of the first length-one CRI. Then, we express the following three lemmata, whose proofs are in the appendix.

<u>Lemma 1</u>

If $\overline{\lim\limits_{M\to\infty}} M\lambda<\lambda^*$, then $\overline{\lim\limits_{M\to\infty}} E\{\ell_{s,M}\}<\infty$

<u>Lemma 2</u>

If $\overline{\lim\limits_{M\to\infty}} M\lambda<\lambda^*$ and $\overline{\lim\limits_{M\to\infty}} M\sigma^2<\infty$, then $\overline{\lim\limits_{M\to\infty}} E\{\ell^2_{s,M}\}<\infty$

<u>Lemma 3</u>

If $M\lambda<1$ , then $H_{1,M} = \pi^{-1}_{1,M}\, E\{\ell_{s,M}\}<\infty$ ; $\forall M$

---

The expected values $E\{\ell_{s,M}\}$ and $E\{\ell^2_{s,M}\}$ are used to bound the expected value $E\{D_{s,M}\}$ in theorem 4. Then, part (i) of the theorem follows from lemmata 1,2, and 3. The specifics are shown in the appendix.

Let us now focus on part (ii) of theorem 4. Given M users in the system, given some algorithm in our class, let $\pi_{d,M}$ denote the steady-state probability defined in the beginning of this subsection, let $\ell_{n,M}$ denote the length of the n-th CRI, and let $\ell_{s,M}$ denote the steady-state CRI length. Let us also define,

$$J(M) \doteq I\, 2^M \quad ; \quad \text{where I is some positive integer.} \qquad (18)$$

Then, we express two lemmata, whose proofs are in the appendix.

<u>Lemma 4</u>

For Poisson users in the system, and any algorithm in the class,

$$\ell_{n,M} \underset{st}{\leq} \ell_{n,2M}$$

Lemma 5

For Poisson users in the system, and any algorithm in the class,

(i) $\lim_{M\to\infty} \pi_{d,J(M)} = \pi_d$ ; that is, the limit converges for every d.

(ii) If, $\lim_{M\to\infty} \sum_{d=1}^{\infty} d\, \pi_{d,J(M)} = E\{\ell_{s,J(M)}\} < \infty$ (19)

then, $\lim_{M\to\infty} \sum_{d=1}^{\infty} d\, \pi_{d,J(M)} = \sum_{d=1}^{\infty} d\, \pi_d$ (20)

---

In the proof of lemma 5, the result in lemma 4 is used. Lemma 5 is finally used to prove part (ii) of theorem 4. The latter step is included in the appendix.

## 5. Performance Evaluation of a Specific Algorithm in the Class

In this section, we study the performance characteristics of a specific simple algorithm, which belongs to the class described in section 2. In particular, the RAA used by the algorithm is either the nondynamic or the dynamic form of Capetanakis' protocol [1]. In the former case, we name the algorithm CCRAR, while in the latter case we name it DCCRAR. In the presence of the limit Poisson user model, the respective throughputs, $\lambda_{\infty}^{*}(1)$ and $\lambda_{\infty}^{*}(2)$, of the above two forms of the algorithm are easily computed, via simple modifications in the equations in [1], where the DCCRAR utilizes a length-$\Delta$ initial arrival interval. It is found that:

$$\lambda_{\infty}^{*}(1) = 0.26$$

$$\lambda_{\infty}^{*}(2) = 0.31, \text{ for } \Delta=4 \qquad (21)$$

In addition, for both the CCRAR and the DCCRAR forms, the quantity $\lambda_2^{*}-\lambda_1^{*}$ in part (ii) of theorem 4 is practically equal to zero. Thus, the throughputs $\lambda_{\infty}^{*}(1)$ and $\lambda_{\infty}^{*}(2)$ are attainable; when the users in the system are Poisson, the limit throughputs of the CCRAR and the DCCRAR are respectively equal to $\lambda_{\infty}^{*}(1)$ and $\lambda_{\infty}^{*}(2)$.

We considered the operation of the CCRAR and the DCCRAR, in the presence of Poisson users. For various numbers, M, of such users, and for various values of the

system input rate, we computed upper and lower bounds, D(U) and D(L), on the corresponding expected per packet delays D. In the computation of those bounds, we utilized the methodology developed in [2]. We omit the specifics of this computation, to avoid redundancy and presentation of tedious but routine expressions. The computed bounds are included in tables 1 and 2, together with the corresponding expected per packet delays induced by TDMA. We note that for the CCRAR, only upper bounds were computed, because the computation of lower bounds presents analytical difficulties. From tables 1 and 2, we observe that for relatively high input rates, TDMA is superior to CCRAR and DCCRAR. As the number of users increases, however, the delays induced by CCRAR and DCCRAR remain uniformly lower than those induced by TDMA, for a high range of input rates. When the number of users tends to infinity (for all practical purposes), so do the expected delays induced by TDMA, while those induced by CCRAR and DCCRAR remain bounded and uniformly good, for input rates in respectively (0, 0.26) and (0, 0.31). We note that when the number of users is 256, the expected delays in tables 1 and 2 are practically the same with those induced when the limit Poisson user model is present.

## 6. Conclusions

The algorithms in this paper are a combination of a random access and a reservation schemes. The head packets in each user queue are involved in collisions for channel reservation, where those collisions are resolved by some random access algorithm with general characteristics. The operation of our algorithms is independent of the number of users in the system, and of the packet generating process per user, as long as the latter is i.i.d. and possesses finite second moment. In addition, when the user population tends to infinity, the throughput of each algorithm in our class is identical to that induced when the limit Poisson user model is present.

We note that for any finite number of users, our algorithms attain the same throughput as TDMA, while the operation of the latter depends on the number of users in the system. Given a small number of users and high input rates, TDMA induces lower

delays than our algorithms do. However, for any arrival rate, as the user population increases to infinity, so do the delays induced by TDMA, while the delays induced by each algorithm in our class remain bounded and uniformly good within the corresponding stability region. In contrast to pure random access schemes, the algorithms in our class attain throughput one, for any finite number of users in the system. In contrast to the ALOHA algorithm in [3], our algorithms are stable even when the user population tends to infinity. Given any finite number of users, our class is also superior to pure random access schemes, in terms of delays.

| M | Input Rate | D(U) | D for TDMA |
|---|---|---|---|
| 4 | 0.010 | 0.31473E+01 | 0.30202E+01 |
| 4 | 0.050 | 0.39284E+01 | 0.31052E+01 |
| 4 | 0.100 | 0.57080E+01 | 0.32222E+01 |
| 4 | 0.150 | 0.10056E+02 | 0.33529E+01 |
| 4 | 0.200 | 0.27744E+02 | 0.34999E+01 |
| 4 | 0.210 | 0.38300E+02 | 0.35316E+01 |
| 4 | 0.230 | 0.97096E+02 | 0.35974E+01 |
| 4 | 0.240 | 0.20678E+03 | 0.36315E+01 |
| 8 | 0.010 | 0.31316E+01 | 0.50404E+01 |
| 8 | 0.050 | 0.38311E+01 | 0.52105E+01 |
| 8 | 0.100 | 0.54327E+01 | 0.54444E+01 |
| 8 | 0.150 | 0.93940E+01 | 0.57058E+01 |
| 8 | 0.200 | 0.26209E+02 | 0.59999E+01 |
| 8 | 0.210 | 0.36721E+02 | 0.60632E+01 |
| 8 | 0.230 | 0.10059E+03 | 0.61948E+01 |
| 8 | 0.240 | 0.24000E+03 | 0.62631E+01 |
| 16 | 0.010 | 0.31237E+01 | 0.90808E+01 |
| 16 | 0.050 | 0.37822E+01 | 0.94210E+01 |
| 16 | 0.100 | 0.52926E+01 | 0.98888E+01 |
| 16 | 0.150 | 0.90489E+01 | 0.10411E+02 |
| 16 | 0.200 | 0.25335E+02 | 0.10999E+02 |
| 16 | 0.210 | 0.35764E+02 | 0.11126E+02 |
| 16 | 0.230 | 0.10226E+03 | 0.11389E+02 |
| 16 | 0.240 | 0.26157E+03 | 0.11526E+02 |
| 32 | 0.010 | 0.31198E+01 | 0.17161E+02 |
| 32 | 0.050 | 0.37576E+01 | 0.17842E+02 |
| 32 | 0.100 | 0.52221E+01 | 0.18777E+02 |
| 32 | 0.150 | 0.88792E+01 | 0.19823E+02 |
| 32 | 0.200 | 0.24868E+02 | 0.20999E+02 |
| 32 | 0.210 | 0.35237E+02 | 0.21253E+02 |
| 32 | 0.230 | 0.10307E+03 | 0.21779E+02 |
| 32 | 0.240 | 0.27407E+03 | 0.22052E+02 |
| 64 | 0.010 | 0.31178E+01 | 0.33323E+02 |
| 64 | 0.050 | 0.37453E+01 | 0.34684E+02 |
| 64 | 0.100 | 0.51866E+01 | 0.36555E+02 |
| 64 | 0.150 | 0.87841E+01 | 0.38647E+02 |
| 64 | 0.200 | 0.24628E+02 | 0.40999E+02 |
| 64 | 0.210 | 0.34961E+02 | 0.41506E+02 |
| 64 | 0.230 | 0.10345E+03 | 0.42558E+02 |
| 64 | 0.240 | 0.28085E+03 | 0.43105E+02 |
| 128 | 0.010 | 0.31168E+01 | 0.65646E+02 |
| 128 | 0.050 | 0.37392E+01 | 0.68368E+02 |
| 128 | 0.100 | 0.51689E+01 | 0.72111E+02 |
| 123 | 0.150 | 0.87390E+01 | 0.76294E+02 |
| 128 | 0.200 | 0.24505E+02 | 0.80999E+02 |
| 128 | 0.210 | 0.34823E+02 | 0.82012E+02 |
| 128 | 0.230 | 0.10367E+03 | 0.84116E+02 |
| 128 | 0.240 | 0.28437E+03 | 0.85210E+02 |
| 256 | 0.010 | 0.31163E+01 | 0.13029E+03 |
| 256 | 0.050 | 0.37360E+01 | 0.13573E+03 |
| 256 | 0.100 | 0.51598E+01 | 0.14322E+03 |
| 256 | 0.150 | 0.87175E+01 | 0.15158E+03 |
| 256 | 0.200 | 0.24442E+02 | 0.16099E+03 |
| 256 | 0.210 | 0.34749E+02 | 0.16302E+03 |
| 256 | 0.230 | 0.10369E+03 | 0.16723E+03 |
| 256 | 0.240 | 0.28600E+03 | 0.16942E+03 |

Table 1

Expected per Packet Delays
Poisson Users - The CCRAR Algorithm

| M | Input Rate | Δ | D(L) | D(U) | D for TDMA |
|---|---|---|---|---|---|
| 4 | 0.050 | 4 | 0.16852E+01 | 0.17049E+01 | 0.31052E+01 |
| 4 | 0.100 | 4 | 0.19654E+01 | 0.20530E+01 | 0.3[illegible]22E+01 |
| 4 | 0.150 | 4 | 0.24170E+01 | 0.26506E+01 | 0.3[illegible]29E+01 |
| 4 | 0.200 | 4 | 0.31962E+01 | 0.37294E+01 | 0.34999E+01 |
| 4 | 0.250 | 4 | 0.47188E+01 | 0.59201E+01 | 0.36666E+01 |
| 4 | 0.300 | 4 | 0.87670E+01 | 0.11920E+02 | 0.38571E+01 |
| 4 | 0.350 | 4 | 0.57818E+02 | 0.85730E+02 | 0.40769E+01 |
| 8 | 0.050 | 4 | 0.17117E+01 | 0.17317E+01 | 0.52105E+01 |
| 8 | 0.100 | 4 | 0.20459E+01 | 0.21373E+01 | 0.54444E+01 |
| 8 | 0.150 | 4 | 0.26231E+01 | 0.28780E+01 | 0.57058E+01 |
| 8 | 0.200 | 4 | 0.37479E+01 | 0.43793E+01 | 0.59999E+01 |
| 8 | 0.250 | 4 | 0.65342E+01 | 0.82228E+01 | 0.63333E+01 |
| 8 | 0.300 | 4 | 0.21595E+02 | 0.29529E+02 | 0.67142E+01 |
| 16 | 0.050 | 4 | 0.17249E+01 | 0.17451E+01 | 0.94210E+01 |
| 16 | 0.100 | 4 | 0.20868E+01 | 0.21[illegible]01E+01 | 0.98888E+01 |
| 16 | 0.150 | 4 | 0.27305E+01 | 0.29964E+01 | 0.10411E+02 |
| 16 | 0.200 | 4 | 0.40535E+01 | 0.47389E+01 | 0.10999E+02 |
| 16 | 0.250 | 4 | 0.77353E+01 | 0.97453E+01 | 0.11666E+02 |
| 16 | 0.300 | 4 | 0.44989E+02 | 0.61639E+02 | 0.12428E+02 |
| 32 | 0.050 | 4 | 0.17316E+01 | 0.17519E+01 | 0.17842E+02 |
| 32 | 0.100 | 4 | 0.21075E+01 | 0.22019E+01 | 0.18777E+02 |
| 32 | 0.150 | 4 | 0.27865E+01 | 0.30582E+01 | 0.19823E+02 |
| 32 | 0.200 | 4 | 0.42219E+01 | 0.49373E+01 | 0.20999E+02 |
| 32 | 0.250 | 4 | 0.84884E+01 | 0.10700E+02 | 0.22333E+02 |
| 32 | 0.300 | 4 | 0.89691E+02 | 0.12301E+03 | 0.23857E+02 |
| 64 | 0.050 | 4 | 0.17349E+01 | 0.17555E+01 | 0.34684E+02 |
| 64 | 0.100 | 4 | 0.21180E+01 | 0.22128E+01 | 0.36555E+02 |
| 64 | 0.150 | 4 | 0.28151E+01 | 0.30897E+01 | 0.38647E+02 |
| 64 | 0.200 | 4 | 0.43105E+01 | 0.50416E+01 | 0.40999E+02 |
| 64 | 0.250 | 4 | 0.89143E+01 | 0.11240E+02 | 0.43666E+02 |
| 64 | 0.300 | 4 | 0.17037E+03 | 0.23379E+03 | 0.46714E+02 |
| 128 | 0.050 | 4 | 0.17366E+01 | 0.17569E+01 | 0.68368E+02 |
| 128 | 0.100 | 4 | 0.21232E+01 | 0.22183E+01 | 0.72111E+02 |
| 128 | 0.150 | 4 | 0.28295E+01 | 0.31056E+01 | 0.76294E+02 |
| 128 | 0.200 | 4 | 0.43560E+01 | 0.50951E+01 | 0.80999E+02 |
| 128 | 0.250 | 4 | 0.91411E+01 | 0.11528E+02 | 0.86333E+02 |
| 128 | 0.300 | 4 | 0.30306E+03 | 0.41598E+03 | 0.92428E+02 |
| 256 | 0.050 | 4 | 0.17374E+01 | 0.17577E+01 | 0.13573E+03 |
| 256 | 0.100 | 4 | 0.21257E+01 | 0.22209E+01 | 0.14322E+03 |
| 256 | 0.150 | 4 | 0.28368E+01 | 0.31137E+01 | 0.15158E+03 |
| 256 | 0.200 | 4 | 0.43789E+01 | 0.51221E+01 | 0.16099E+03 |
| 256 | 0.250 | 4 | 0.92579E+01 | 0.11675E+02 | 0.17166E+03 |
| 256 | 0.300 | 4 | 0.49121E+03 | 0.67432E+03 | 0.18385E+03 |

Table 2

Expected per Packet Delays
Poisson Users - The DCCRAR Algorithm

## Appendix

### Proof of Theorem 1

From (1), we have $L_k<\infty$; $\forall k<\infty$. Then, from (3) we easily conclude that,

$$\lim_{d\to\infty} d^{-1} L_d = M\lambda \tag{A.1}$$

which implies the following: Given $\varepsilon>0$, there exists $d_o$, such that:

$$L_d \leq (M\lambda + \varepsilon)d \quad ; \forall d\geq d_o \tag{A.2}$$

Let $N$ denote the set of natural numbers, and let us define,

$$B \stackrel{\Delta}{=} \max(0, L_1-(M\lambda + \varepsilon), \ldots, L_k-k(M\lambda + \varepsilon), \ldots, L_{d_o-1}-(d_o-1)(M\lambda + \varepsilon))$$

Then, due to (A.2) we conclude,

$$L_d \leq (M\lambda + \varepsilon)d + B \quad ; \quad \forall d\varepsilon N \tag{A.3}$$

$$E\{C_{k+1}\} = E\{L_d\} \leq (M\lambda + \varepsilon) \; E\{C_k\} + B \quad ; \quad \forall k\geq 1 \tag{A.4}$$

Let $\alpha \stackrel{\Delta}{=} M\lambda + \varepsilon<1$. Then, (A.4) gives,

$$E\{C_{k+1}\} \leq \alpha^k \; E\{C_1\} + B\,\frac{1-\alpha^k}{1-\alpha} = \alpha^k + B\,\frac{1-\alpha^k}{1-\alpha} <$$

$$< 1 + \frac{B}{1-\alpha} < \infty \quad ; \quad \forall k \tag{A.5}$$

Since, given any $\varepsilon>0$, such that $M\lambda + \varepsilon<1$, we have that $E\{C_{k+1}\} < \infty$ ; $\forall k$, then we conclude that the chain $\{C_i\}_{i\geq 1}$ is ergodic, if $M\lambda<1$.

### Proof of Theorem 2

Let us consider some CRI, and let there be some packet arrival within the time interval that corresponds to its length. Let $\omega$ be the length between the arrival instant of the packet and the end of the CRI. Let $\theta$ be the length between the ending point of the CRI, and the time instant when the successful transmission of the packet just ends. Let us then define.

$\psi_d$ : The sum of the lengths $\omega$ of all packet arrivals within some CRI, given that the length of the CRI is d.

$z_d$ : The sum of the lengths $\theta$ of all packet arrivals within some CRI, given that the length of the CRI is d.

$p(\ell|d)$ : Given that some CRI has length d, the probability that the next CRI has length $\ell$.

$h_d$ : The length between the end of some CRI, and the end of the first after that length-one CRI, given that the length of the former is d.

$w_d$ : The cumulative delay of all the packets that are successfully transmitted within the time interval that corresponds to $h_d$.

$\Psi_d = E\{\psi_d\}$ , $W_d = E\{w_d\}$ , $H_d = E\{h_d\}$ , $Z_d = E\{z_d\}$

From the operation of the algorithms, we then easily conclude that the following equations hold.

$$H_d = \sum_{\ell=1}^{\infty} \ell \; p(\ell|d) + \sum_{\ell=2}^{\infty} H_\ell \; p(\ell|d) \qquad \text{(A.6)}$$

$$W_d = Z_d + \Psi_d + \sum_{\ell=2}^{\infty} W_\ell \; p(\ell|d) \qquad \text{(A.7)}$$

Using the theory of infinite dimensionality linear systems, we can express the following theorem, whose proof is identical to the proofs of the parallel theorems in [2], and is omitted.

Theorem A

Let $M\lambda<1$. Then, the system in (A.6) has a nonnegative solution, which is also unique within the class of quadratically bounded sequences. Let in addition the packet generating process per user have finite second moment. Then, the system in (A.7) has a nonnegative solution, which is also unique within the class of quadratically bounded sequences.

---

Since W in (5) and H in (8) are respectively $W_1$ and $H_1$, the conditions in theorem A also guarantee boundness of H and W.

Proof of Lemma 1

Clearly, $E\{\ell_{s,M}\} = \sum_{d=1}^{\infty} d\,\pi_{d,M}$. On the other hand, from (A.5) we conclude that for every M and for $M\lambda<1$, there exists $b_M<\infty$, such that, $E\{C_k\}<b_M$ ; $\forall k$. Then, by Fatou's lemma we conclude that,

$$E\{\ell_{s,M}\} < b_M < \infty \ ; \ \forall M, \text{ and } M\lambda = \delta<1 \qquad \text{(A.8)}$$

Now, from the condition $\overline{\lim}_{k\to\infty} k^{-1} L_k < \infty$ in (1), we conclude that there exist finite positive constants c and p, such that $\overline{\lim}_{k\to\infty} k^{-1} L_k = c$, and such that, for every $\varepsilon>0$, we have,

$$L_k \le (c+\varepsilon)\, k+p \quad ; \quad \forall\, k \qquad \text{(A.9)}$$

Substituting (A.9) in (3), and for $p_0$ being as in section 4, we obtain,

$$L_d \le (c+\varepsilon+1)M\lambda d + p \qquad \text{(A.10)}$$

Thus,

$$E\{\ell_{s,M}\} = \sum_{d=1}^{\infty} L_d\, \pi_{d,M} \le (c+\varepsilon+1)M\lambda \sum_{d=1}^{\infty} d\pi_{d,M} + p =$$

$$= (c+\varepsilon+1)M\lambda\ E\{\ell_{s,M}\} + p \qquad \text{(A.11)}$$

; where $M\lambda = \delta$. Provided that $\delta<(c+\varepsilon+1)^{-1}$, (A.11) gives,

$$E\{\ell_{s,M}\} \le p[1-(c+\varepsilon+1)M\lambda]^{-1} = p[1-(\varepsilon+c+1)\delta]^{-1} \ ; \ \forall M,\ \forall \varepsilon>0:$$
$$: \ \delta<(c+\varepsilon+1)^{-1} \qquad \text{(A.12)}$$

Then,

$$\overline{\lim_{M\to\infty}}\ E\{\ell_{s,M}\} \le p[1-(c+1)^{-1}] \quad ; \quad \text{if } \overline{\lim_{M\to\infty}}\ M\lambda < (c+1)^{-1}$$

Proof of Lemma 2

Let $p(k|d)$ be as in (4), let $L_k$ and $L_k^{(2)}$ be as in (1), and let us define, $L_d^{(2)} = E\{C_{i+1}^2 \mid C_i = d\}$ ; $\forall i$. Then, given M, the inequality (A.14) below can be easily established, where $p_0$ is as in section 4, and where,

$$M(1-p_0) < M\lambda \tag{A.13}$$

$$L_d^{(2)} \leq \sum_{k=0}^{M} L_k^{(2)} \; p(k|d) + \sigma^2 [1-p_0^d]^{-1} \sum_{k=0}^{M} k^2 p(k|d) +$$

$$+ 2\lambda d \; [1-p_0^d]^{-1} \sum_{k=0}^{M} k \; L_k \; p(k|d) \tag{A.14}$$

At the same time, we clearly have,

$$E\{\ell^2_{s,M}\} = \sum_{d=1}^{\infty} L_d^{(2)} \pi_{d,M} = \sum_{d=1}^{\infty} d^2 \; \pi_{d,M} \tag{A.15}$$

(i) From the condition $\overline{\lim_{k\to\infty}} \, k^{-2} \, L_k^{(2)} < \infty$ in (1), and for $M\lambda < 1$, we can show, in parallel to the proof of theorem 1, that there exists finite constant $b_M$, such that $E\{C_k^2\} < b_M$ ; $\forall k$, $\forall M$. Then, by Fatou's lemma we conclude,

$$E\{\ell^2_{s,M}\} < b_M < \infty \; ; \quad \forall \; M \text{ and } M\lambda = \delta < 1 \tag{A.16}$$

(ii) From the condition $\overline{\lim_{k\to\infty}} \, k^{-2} \, L_k^{(2)} < \infty$, we conclude that there exist finite positive constants $\alpha, \beta$, and $\gamma$, such that, $\overline{\lim_{k\to\infty}} \, k^{-2} \, L_k^{(2)} = \alpha$, and such that for every $\varepsilon > 0$ we have,

$$L_k^{(2)} \leq (\alpha+\varepsilon) k^2 + \beta k + \gamma \; ; \quad \forall k \tag{A.17}$$

Substituting the bounds in (A.9) and (A.17), in inequality (A.14), we obtain,

$$L_d^{(2)} < (\alpha+\varepsilon)[M(1-p_0^d)]^2 + [M(1-p_0^d)][\beta + M\sigma^2 + 2cd \; M\lambda] +$$

$$+ 2dM\lambda + \gamma \quad ; \quad \forall \varepsilon > 0 \tag{A.18}$$

In addition, we have,

$$1-p_0^d < (1-p_0) \; d \quad , \quad M(1-p_0) < M\lambda \tag{A.19}$$

From (A.18) and (A.19), we obtain,

$$L_d^{(2)} < (M\lambda)^2 [\alpha+\varepsilon+2c] d^2 + M\lambda \; [\beta+M\sigma^2 + 2] d + \gamma \; ; \quad \forall \varepsilon > 0 \tag{A.20}$$

Due to (1.a), $\exists \, \varepsilon > 0$: $M\lambda = \delta < [\varepsilon+\alpha+2c]^{-1/2}$, and expressions (A.15) and (A.20), in conjuction with (A.12), give,

$$E\{\ell^2_{s,M}\} < [1-\delta^2[\varepsilon+\alpha+2c)]^{-1} [\delta(\beta+M\sigma^2 + 2) E\{\ell_{s,M}\} + \gamma]$$

$$\leq [1-\delta^2(\varepsilon+\alpha+2c)]^{-1} \{\delta[\beta+M\sigma^2 + 2] [1-(c+1)\delta]^{-1} + \gamma\} \; ; \; \forall M, \forall \varepsilon>0 \tag{A.21}$$

From $\overline{\lim_{M\to\infty}} M\sigma^2<\infty$, from (A.16), and from (A.21), we conclude that, there exists some bounded constant B, such that,

$$E\{\ell^2_{s,M}\} \leq B \quad ; \quad \forall M$$

Proof of Lemma 3

As in [2], it can be shown that there exist positive bounded constants $\alpha_M$ and $\beta_M$, such that,

$$H_{d,M} \leq \alpha_M d + \beta_M \quad ; \quad \forall d \quad ; \forall M$$

Then, from (A.8), we have that, $E\{\ell_{s,M}\} < b_M < \infty \;\; ; \;\; \forall M$, and

$$H_M \triangleq \sum_{d=1}^{\infty} H_{d,M}\,\pi_{d,M} \leq \alpha_M E\{\ell_{s,M}\} + \beta_M < \alpha_M b_M + \beta_M < \infty \quad ; \quad \forall M \tag{A.22}$$

In addition,

$$H_M = \sum_{d=1}^{\infty} L_d\,\pi_{d,M} + \sum_{d=1}^{\infty} \sum_{\ell=2}^{\infty} H_{\ell,M}\, p(\ell|d)\pi_{d,M} =$$

$$= E\{\ell_{s,M}\} + \sum_{\ell=2}^{\infty} H_{\ell,M} \sum_{d=1}^{\infty} p(\ell|d)\,\pi_{d,M} = E\{\ell_{s,M}\} + \sum_{\ell=2}^{\infty} H_{\ell,M}\,\pi_{\ell,M}$$

$$= E\{\ell_{s,M}\} + H_M - \pi_{1,M} H_{1,M} \quad ; \quad \forall M$$

Since $H_M$ is finite by (A.22), the result follows from the latter equation.

Proof of Theorem 4 - Part (i)

Consider the quantities defined in the proof of theorem 2. Given M users, we will denote the arbitrary such quantity $B_{d,M}$ (instead of $B_d$). For $M\lambda < 1$, using the result in lemma 3, and following the same approach as in its proof, we obtain,

$$E\{\mathcal{D}_{s,M}\} = [M\lambda\ E\{\ell_{s,M}\}]^{-1} \sum_{d=1}^{\infty} [Z_{d,M} + \Psi_{d,M}]\ \pi_{d,M} \qquad (A.23)$$

; where for c and p as in (A.9), for $p_0$ as in section 4, and for $\sigma^2$ as in theorem 4, we have,

$$Z_{d,M} = cM[1-p_0]\ M\lambda d^2 + p[M\lambda + M(1-p_0)(M\sigma^2)]\ d \qquad (A.24)$$

$$\Psi_{d,M} = 2^{-1}(M\lambda)\ d^2 \qquad (A.25)$$

Substituting expressions (A.24) and (A.25) in (A.23), we obtain,

$$E\{\mathcal{D}_{s,M}\} = [M\lambda\ E\{\ell_{s,M}\}]^{-1}\ \{2^{-1}M\lambda + cM(1-p_0)\ M\lambda\ E\{\ell^2_{s,M}\} +$$

$$+ [M\lambda + M(1-p_0)(M\sigma^2)]\ E\{\ell_{s,M}\}\} \qquad (A.26)$$

Using the inequalities in (A.19) and lemmata 1 and 2, we conclude from (A.26) that, for $\overline{\lim_{M\to\infty}}\ M\lambda < \lambda_1^*$ and $\overline{\lim_{M\to\infty}}\ M\sigma^2 < \infty$, there exists some bounded constant $\gamma$, such that, $E\{\mathcal{D}_{s,M}\} < \gamma$ ; $\forall M$. Then, $\overline{\lim_{M\to\infty}}\ E\{\mathcal{D}_{s,M}\} < \infty$ follows.

We note, that from (A.23), the following inequality evolves easily:

$$E\{\mathcal{D}_{s,M}\} \geq 2^{-1}\ E\{\ell_{s,M}\} \qquad (A.27)$$

Proof of Lemma 4

Given some algorithm in the class, let $\ell_k$ denote the length in (1), as induced by the RAA of the algorithm. Let us define,

$P_M(k|N)$ : Given M users, given N total arrivals, the probability that k users are active

$$P_M(k\leq t|N) = \sum_{i=0}^{t} P_M(i|N)$$

$P_M(N|d)$ : Given M users, given some CRI of length d, the probability that N total arrivals occur within the arrival interval that corresponds to the CRI.

We first state and prove a proposition:

Proposition A

Given Poisson users, we have,

$$P_M(k\le t|N) \ge P_{2M}(k\le t|N) \ ; \ \forall t \qquad (A.28)$$

Proof

When the users are independent and Poisson, the probability $P_M(k\le t|N)$ is the same with the probability of placing N objects in at most t cells of an M-cell capacity buffer, when each of the N objects is placed independently and with probability $M^{-1}$ per cell. Then, (A.28) follows trivially.

---

We can now express the following equation, for x being a nonnegative integer.

$$\Pr(\ell_{n,M}\le x|\ell_{n-1,M}=d) = \begin{cases} \Pr(\ell_0=0)\ P_M(0|d) = 0 \ ; \ x = 0 \\ P_M(0|d)\ \Pr(\ell_0\le x) + \\ +\sum_{N=1}^{x} P_M(N|d) \sum_{k=1}^{\min(M,N)} P_M(k|N)\ \Pr(\ell_k \le x-N) \ ; \ x\ge 1 \end{cases} \qquad (A.29)$$

Due to (A.28), due to $\ell_k \underset{st}{\le} \ell_{k+1}$ ; $\forall k$ in (1), and from (A.29), we easily conclude,

$$\Pr(\ell_{n,M}\le x|\ell_{n-1,M}=d) \ge \Pr(\ell_{n,2M}\le x|\ell_{n-1,2M}=d) \qquad ; \forall x, \forall d$$

We now note that $P_M(k\le x|N)$ and $\Pr(\ell_k\le x-N)$ are both decreasing with increasing N. Then, in conjuction with (1.b) we conclude that $\sum_k P_M(k|N)\Pr(\ell_k\le x-N)$ is also decreasing with increasing N. In addition, $P_M(N\le \xi|d)$ is decreasing with increasing d. Due to the above, we observe that $\Pr(\ell_{n,M}\le x|\ell_{n,M-1}=d)$ is decreasing with increasing d, as well. Let us now hypothesize that $\ell_{n,M} \underset{st}{\le} \ell_{n,2M}$, for some n. Then, $\Pr(\ell_{n+1,M}\le x) =$

$$= \sum_d \Pr(\ell_{n+1,M}\le x|\ell_{n,M}=d)\cdot \Pr(\ell_{n,M}=d) \ge$$

$$\ge \sum_d \Pr(\ell_{n+1,2M}\le x|\ell_{n,2M}=d)\ \Pr(\ell_{n,M}=d) \ge$$

$$\geq \sum_{d} \Pr(\ell_{n+1,2M} \leq x \mid \ell_{n,2M} = d) \Pr(\ell_{n,2M} = d) = \Pr(\ell_{n+1,2M} \leq x)$$

The last inequality follows by hypothesis. Since $\ell_{1,M} = \ell_{1,2M} = 1$, and by induction, we obtain the result in the lemma; that is,

$$\Pr(\ell_{n,M} \leq x) \geq \Pr(\ell_{n,2M} \leq x) \quad ; \forall x$$

Proof of Lemma 5

(i) Due to the result in lemma 4, we easily conclude that, $\pi_{1,J(M)} \geq \pi_{1,J(M+1)}$. Since $\pi_{1,J(M)}$ is bounded from below, we also conclude,

$$\lim_{M\to\infty} \pi_{1,J(M)} = \pi_1 < \infty \qquad \text{(A.30)}$$

From lemma 4, we also conclude,

$$\sum_{d=1}^{x+1} \pi_{d,J(M)} \geq \sum_{d=1}^{x+1} \pi_{d,J(M+1)} \qquad \text{(A.31)}$$

Due to (A.31), we have,

$$\lim_{M\to\infty} \sum_{d=1}^{x+1} \pi_{d,J(M)} = \lim_{M\to\infty} (\pi_{x+1,J(M)} + \sum_{d=1}^{x} \pi_{d,J(M)}) = I \qquad \text{(A.32)}$$

Due to (A.32), and by induction, we easily conlude that $\pi_{x,J(M)}$ converges, for every x; that is,

$$\lim_{M\to\infty} \pi_{x,J(M)} = \pi_x < \infty \quad ; \quad \forall x \qquad \text{(A.33)}$$

(ii) For arbitrary n, we have,

$$\sum_{d=1}^{\infty} d\, \pi_{d,J(M)} \geq \sum_{d=n+1}^{\infty} d\, \pi_{d,J(M)} \geq n[1-\sum_{d=1}^{n} \pi_{d,J(M)}] \qquad \text{(A.34)}$$

Let us now assume that $\sum_{d=1}^{\infty} \pi_d = \mu < 1$. Then, we have that given $\epsilon > 0$ such that $\mu + \epsilon < 1$, given n, there exists $M_0$, such that, $\sum_{d=1}^{n} \pi_{d,J(M)} < \mu + \epsilon \; ; \forall M > M_0$. Then, (A.34) gives $\sum_{d=1}^{\infty} d\, \pi_{d,J(M)} > n[1-(\mu+\epsilon)] \; ; \forall M > M_0$. Since the above holds for

arbitrary n, we then conclude, $\lim_{M\to\infty} \sum_{d=1}^{\infty} d\, \pi_{d,J(M)} = \infty$. Thus, if (19) holds, then $\sum_{d=1}^{\infty} \pi_d \leq 1$ presents a contradiction. Therefore, subject to (19), we have:

$$\sum_{d=1}^{\infty} \pi_d = 1 \qquad \text{(A.35)}$$

Due to (A.31) and (A.35), we conclude,

$$\sum_{d=n}^{\infty} \pi_d \geq \sum_{d=n}^{\infty} \pi_{d,J(M)} \quad ; \quad \forall n,\ \forall M \qquad \text{(A.36)}$$

Applying (A.36), we obtain,

$$\sum_{d=1}^{\infty} d\pi_{d,J(M)} = \sum_{n=1}^{\infty} \sum_{d=n}^{\infty} \pi_{d,J(M)} \leq \sum_{n=1}^{\infty} \sum_{d=n}^{\infty} \pi_d = \sum_{d=1}^{\infty} d\pi_d \quad ; \quad \forall M \qquad \text{(A.37)}$$

In addition,

$$\sum_{d=1}^{N} d\pi_{d,J(M)} \leq \sum_{d=1}^{\infty} d\pi_{d,J(M)} \leq \sum_{d=1}^{\infty} d\pi_{d,J(M+1)} \leq \sum_{d=1}^{\infty} d\pi_d \quad ; \quad \forall N$$

$$\lim_{M\to\infty} \sum_{d=1}^{N} d\pi_{d,J(M)} = \sum_{d=1}^{N} d\pi_d \leq \lim_{M\to\infty} \sum_{d=1}^{\infty} d\pi_{d,J(M)} \leq \sum_{d=1}^{\infty} d\pi_d \quad ; \quad \forall N \qquad \text{(A.38)}$$

From (A.37) and (A.38), we conclude (20).

Proof of Theorem 4 - Part (ii)

The condition $\overline{\lim}_{k\to\infty} k^{-1} L_k < \infty$ in (1), implies $\underline{\lim}_{k\to\infty} k^{-1} L_k < \infty$. The latter means that there exists some finite positive constant $\eta$, such that, $\underline{\lim}_{k\to\infty} k^{-1} L_k = \eta$. Thus, given [illegible], there exists $k_0(\cdot)$, such that,

$$[illegible] \quad ; \quad \forall\ [illegible]\ k_0(\cdot) \triangleq k_0 \qquad \text{(A.39)}$$

[illegible] define,

$$[illegible] \max [illegible] - L_k) \qquad \text{(A.40)}$$

$$[illegible] \qquad \text{(A.41)}$$

We note that $\eta < c$, where c is as in (A.9). We now state the following theorem.

Theorem A

Consider Poisson users. If there are M such users in the system, let $\lambda_M$ be the Poisson intensity per user. As M varies, let the product $M\lambda_M$ remain fixed and equal to $\delta$. If, for $\varepsilon$ as in (A.41),

$$\delta > \lambda^* + \varepsilon, \text{ then } \overline{\lim_{M\to\infty}} \; E\{\ell_{s,M}\} = \infty.$$

Proof

We will prove the theorem by contradiction. Let $\overline{\lim_{M\to\infty}} E\{\ell_{s,M}\} < \infty$, and let the quantities $L_d$ and $p(k|d)$ in (3) and (4), be denoted instead $L_{d,M}$ and $p_M(k|d)$, to indicate explicitly the number of users in the system. Then, considering (A.39) and (A.40), and for $p_0$ in (4) equal to $\exp\{-\lambda_M\}$ here, when the user population is M, we obtain,

$$L_{d,J(M)} = \sum_{k=0}^{J(M)} L_k \, p_M(k|d) \geq (\eta-\zeta)[1-\exp\{\frac{-\delta d}{J(M)}\}] -$$

$$- \theta \; [1-\exp\{-\frac{\delta d}{J(M)}\}] \qquad \text{(A.41)}$$

; where J(M) is as in (18).

Let us now select $\zeta_1 : \eta(\delta-\zeta_1) > 1$. Then, there exists $\alpha$, such that,

$$1-\exp\{-\frac{\delta d}{J(M)}\} \geq (\delta-\zeta_1)\frac{d}{J(M)} \; ; \; \forall \; d : 0 \leq d \leq \alpha \; J(M) \qquad \text{(A.42)}$$

Then, (A.41) gives:

$$L_{d,J(M)} \geq \eta(\delta-\zeta_1)d - \theta[1-\exp\{-\frac{\delta d}{J(M)}\}] \qquad \text{(A.43)}$$

and thus,

$$E\{\ell_{s,J(M)}\} \geq \eta(\delta-\zeta_1)\sum_{d=1}^{\alpha J(M)} d\pi_{d,J(M)} - \theta \sum_{d=1}^{\infty} \pi_{d,J(M)}[1-\exp\{-\frac{\delta d}{J(M)}\}]$$

$$= \eta(\delta-\zeta_1)[E\{\ell_{s,J(M)}\} - \sum_{d=\alpha J(M)+1}^{\infty} d\pi_{d,J(M)}] - \theta \sum_{d=1}^{\infty} \pi_{d,J(M)} \; [1-\exp\{-\frac{\delta d}{J(M)}\}]$$

(A.44)

But,

$$\overline{\lim_{M\to\infty}} \sum_{d=\alpha J(M)+1}^{\infty} d\pi_{d,J(M)} \leq \overline{\lim_{M\to\infty}} \sum_{d=c}^{\infty} d\pi_{d,J(M)} \quad ; \quad \forall c \tag{A.45}$$

In addition, since by hypothesis $\lim\limits_{M\to\infty} E\{\ell_{s,J(M)}\} < \infty$, and from lemma 5, we have that, given $\nu>0$, there exists $c(\nu)$, such that,

$$\overline{\lim_{M\to\infty}} \sum_{d=c}^{\infty} d\pi_{d,J(M)} < \nu \quad ; \quad \forall\, c \geq c(\nu) \tag{A.46}$$

From (A.45) and (A.46), we conclude,

$$\overline{\lim_{M\to\infty}} \sum_{d=\alpha J(M)+1}^{\infty} d\pi_{d,J(M)} = 0 \tag{A.47}$$

and thus,

$$\lim_{M\to\infty} \sum_{d=\alpha J(M)+1}^{\infty} d\pi_{d,J(M)} = 0 \tag{A.48}$$

It can be seen that,

$$\lim_{M\to\infty} \sum_{d=1}^{\infty} \pi_{d,J(M)} \left[1-\exp\{-\frac{\delta d}{J(M)}\}\right] \leq \sum_{d=N+1}^{\infty} \pi_d \quad ; \quad \forall N$$

Therefore, due to (A.47), we conclude,

$$\lim_{M\to\infty} \sum_{d=1}^{\infty} \pi_{d,J(M)} \left[1-\exp\{-\frac{\delta d}{J(M)}\}\right] = 0 \tag{A.49}$$

Then, from (A.44), (A.48), and (A.49), we obtain,

$$\lim_{M\to\infty} E\{\ell_{s,J(M)}\} \geq \eta(\delta-\zeta_1) \lim_{M\to\infty} E\{\ell_{s,J(M)}\} \tag{A.50}$$

But, since $\eta(\delta-\zeta_1) > 1$, (A.50) contradicts the hypothesis. Thus, $\overline{\lim\limits_{M\to\infty}} E\{\ell_{s,M}\} = \infty$, and the proof of the theorem is complete.

Due to the inequality in (A.27), we have,

$$\overline{\lim_{M\to\infty}} E\{\mathcal{D}_{s,M}\} \geq 2^{-1} \overline{\lim_{M\to\infty}} E\{\ell_{s,M}\} \tag{A.51}$$

Due to (A.51) and theorem A, we conclude that for any $\delta>\lambda_2^*$, $\overline{\lim\limits_{M\to\infty}} E\{\mathcal{D}_{s,M}\} = \infty$, and the proof of part (ii) in theorem 4 is now complete.

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